III Workshop of DGFM/SMF, Nov. 28 -Dec. 3/99, León, Gto., Mexico, [gr-qc/9912nnn]

AN ERMAKOV STUDY OF $Q \neq 0$ EFRW MINISUPERSPACE OSCILLATORS

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Abstract. A previous work on the Ermakov approach for empty FRW minisuperspace universes of Hartle-Hawking factor ordering parameter $Q = 0$ is extended to the $Q \neq 0$ cases.

In a previous work, we presented the EFRW (empty FRW) minisuperspace oscillator of $Q = 0$ Hartle-Hawking factor ordering parameter in the Ermakov framework [1]. Here, we apply the Ermakov procedure to the same cosmological oscillators for $Q \neq 0$.

The EFRW Wheeler-DeWitt (WDW) minisuperspace equation reads

$$\frac{d^2\Psi}{d\Omega^2} + Q\frac{d\Psi}{d\Omega} - \kappa e^{-4\Omega}\Psi(\Omega) = 0 \; , \tag{1}$$

where Q will be considered as a free parameter, the variable Ω is Misner's time, and κ is the curvature index of the FRW universe; $\kappa = 1, 0, -1$ for closed, flat, open universes, respectively. For $\kappa = \pm 1$ the general solution is expressed in terms of Bessel functions,

$$\Psi_\alpha^+(\Omega) = e^{-2\alpha\Omega}\left(C_1 I_\alpha(\frac{1}{2}e^{-2\Omega}) + C_2 K_\alpha(\frac{1}{2}e^{-2\Omega})\right) \tag{2}$$

and

$$\Psi_\alpha^-(\Omega) = e^{-2\alpha\Omega}\left(C_1 J_\alpha(\frac{1}{2}e^{-2\Omega}) + C_2 Y_\alpha(\frac{1}{2}e^{-2\Omega})\right) , \tag{3}$$

respectively, where $\alpha = Q/4$. The case $\kappa = 0$ does not correspond to a parametric oscillator and will not be dealt with here. Eq. (1) can be mapped in a known way to the canonical equations for a classical point particle of mass $M = e^{Q\Omega}$, generalized coordinate $q = \Psi$, momentum $p = e^{Q\Omega}\dot\Psi$, (i.e., velocity $v = \dot\Psi$), and identifying Misner's time Ω with the classical Hamiltonian time. Thus, one is led to

$$\dot q \equiv \frac{dq}{d\Omega} \;\; = \;\; e^{-Q\Omega}p \tag{4}$$

$$\dot{p} \equiv \frac{dp}{d\Omega} \;=\; \kappa e^{(Q-4)\Omega} q \;. \tag{5}$$

These equations describe the canonical motion for a classical EFRW point universe as derived from the 'time'-dependent Hamiltonian

$$H_{\text{cl}}(\Omega) = e^{-Q\Omega}\frac{p^2}{2} - \kappa e^{(Q-4)\Omega}\frac{q^2}{2} \;. \tag{6}$$

The Ermakov invariant $\mathcal{I}(\Omega)$, which is algebraically built as a constant of motion, can be written (for details see [2])

$$\mathcal{I}(\Omega) = (\rho^2) \cdot \frac{p^2}{2} - (e^{Q\Omega}\rho\dot{\rho}) \cdot pq + (e^{2Q\Omega}\dot{\rho}^2 + \frac{1}{\rho^2}) \cdot \frac{q^2}{2} \;, \tag{7}$$

where ρ is the solution of the Milne-Pinney (MP) equation, $\ddot{\rho}+Q\dot{\rho}-\kappa e^{-4\Omega}\rho = \frac{e^{-2Q\Omega}}{\rho^3}$. There is a well-defined prescription introduced by Pinney in 1950 of writing ρ as a function of the particular solutions of the corresponding parametric oscillator problem, i.e., the modified Bessel functions in the EFRW case. In terms of the function $\rho_\pm(\Omega)$ the EFRW Ermakov invariant reads

$$\mathcal{I}_{EFRW}^{\pm} = \frac{(\rho_\pm p - e^{Q\Omega}\dot{\rho}_\pm q)^2}{2} + \frac{q^2}{2\rho_\pm^2} = \frac{e^{2Q\Omega}}{2}\left(\rho_\pm\dot{\Psi}_\alpha^{\pm} - \dot{\rho}_\pm\Psi_\alpha^{\pm}\right)^2 + \frac{1}{2}\left(\frac{\Psi_\alpha^{\pm}}{\rho_\pm}\right)^2 . \tag{8}$$

The calculation of $\mathcal{I}_{\text{EFRW}}^{\pm}$ has been carried by using linear combinations of Bessel functions [3] respecting the initial conditions for the MP function as given by Eliezer and Gray [4].

As in our previous paper, we calculate the 'time'-dependent generating function $S(q,\mathcal{I},\Omega)$ allowing one to pass to new canonical variables for which $\mathcal{I}$ is chosen to be the new "momentum"

$$S(q,\mathcal{I},\Omega) = e^{Q\Omega}\frac{q^2}{2}\frac{\dot{\rho}}{\rho} + \mathcal{I}\arcsin\left[\frac{q}{\sqrt{2\mathcal{I}\rho^2}}\right] + \frac{q\sqrt{2\mathcal{I}\rho^2 - q^2}}{2\rho^2} \;, \tag{9}$$

where we have put to zero the constant of integration. The canonical variables are now $q_1 = \rho\sqrt{2\mathcal{I}}\sin\theta$ and $p_1 = \frac{\sqrt{2\mathcal{I}}}{\rho}\left(\cos\theta + e^{Q\Omega}\dot{\rho}\rho\sin\theta\right)$. The dynamical angle will be $\Delta\theta^d = \int_{\Omega_0}^{\Omega}\langle\frac{\partial H_{\text{new}}}{\partial\mathcal{I}}\rangle d\Omega' = \int_0^{\Omega}[\frac{e^{-Q\Omega'}}{\rho^2} - \frac{1}{2}\frac{d}{d\Omega'}(e^{Q\Omega'}\dot{\rho}\rho) + e^{Q\Omega'}\dot{\rho}^2]d\Omega'$, whereas the geometrical angle reads $\Delta\theta^g = \frac{1}{2}\int_{\Omega_0}^{\Omega}[\frac{d}{d\Omega'}(e^{Q\Omega'}\dot{\rho}\rho) - 2e^{Q\Omega'}\dot{\rho}^2]d\Omega'$. Thus, the total change of angle is $\Delta\theta = \int_{\Omega_0}^{\Omega}\frac{e^{-Q\Omega'}}{\rho^2}d\Omega'$. On the Misner time axis, going to $-\infty$ means going to the origin of the universe, whereas $\Omega_0 = 0$ means the present epoch. Using these cosmological limits one can see that the total change of angle $\Delta\theta$ during the cosmological evolution in Ω time is up to a sign the Laplace transform of parameter Q of the inverse square of the MP function, $\Delta\theta = -L_{1/\rho^2}(Q)$.

We end up with the possible interpretation of the Ermakov invariant in empty minisuperspace cosmology. If one makes an adiabatic series expansion of the invariant, the leading term that defines the adiabatic regime gives the number of created 'quanta' and there were authors giving classical descriptions of the cosmological particle production in such terms [5]. On the other hand, a simple physical interpretation of the Ermakov invariant has

been introduced by Eliezer and Gray, who associated it with the angular momentum of the so-called 2D auxiliary motion [4]. Thus, we claim that in EFRW minisuperspace cosmologies the classical Ermakov invariant gives the number of auxiliary angular momentum adiabatic excitations with which the universe is created at the initial singularity.

In conclusion, we extended here our previous Ermakov procedure for the EFRW WDW parametric equation from $Q = 0$ to the $Q \neq 0$ cases. Plots of the relevant quantities are displayed in the following.
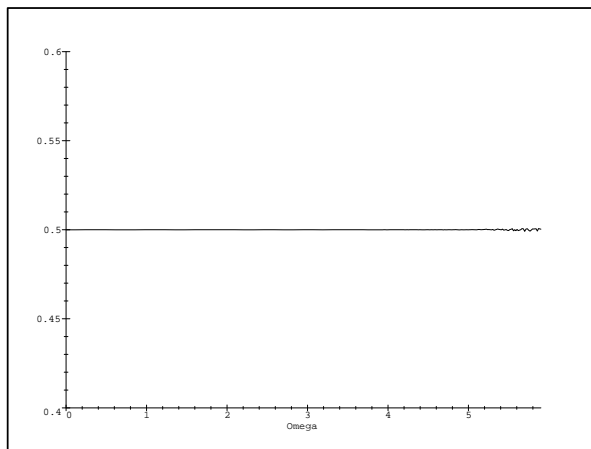
Fig. 1: $\mathcal{I}_{EFRW}^{+}(\Omega)$ for the $Q = 3$ case, corresponding to a singularity with one auxiliary angular momentum excitation.
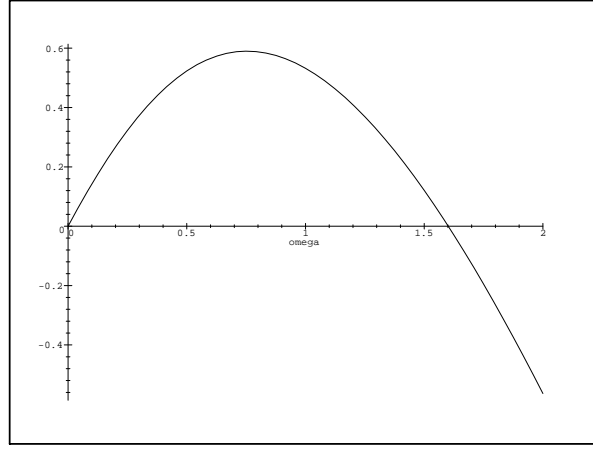
Fig. 2: The dynamical angle as a function of Ω for the closed EFRW model and $Q = 1$.
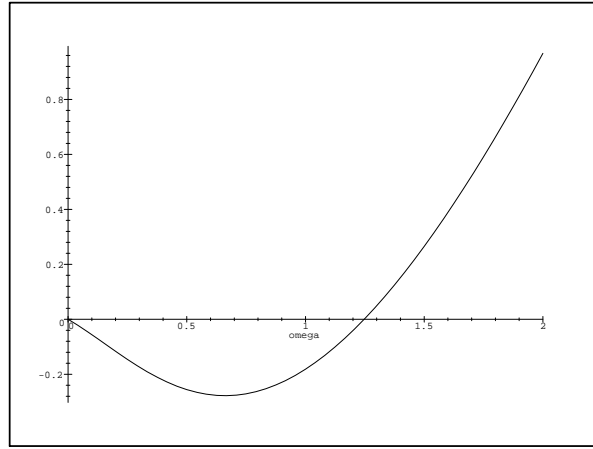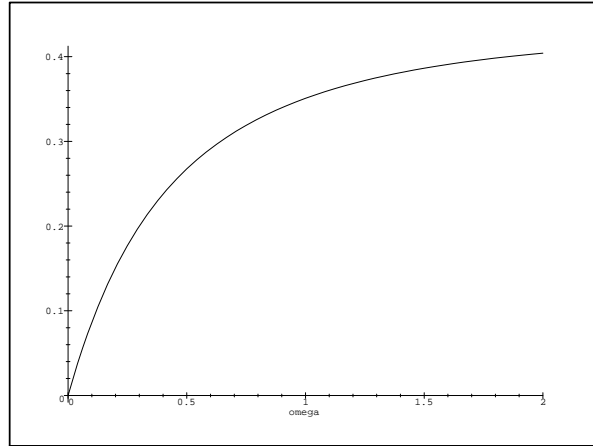

Fig. 3: The geometrical angle for the same case.

Fig. 4: The total angle as a function of Ω for the same case.

References

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